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Differential Geometry of the Vortex Filament Equation

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Abstract. Differential calculus on the space of asymptotically linear curves is developed. The calculus is applied to the vortex filament equation in its Hamiltonian description. The recursion operator generating the infinite sequence of commuting flows is shown to be hereditary. The system is shown to have a description with a Hamiltonian pair. Master symmetries are found and are applied to deriving an expression of the constants of motion in involution. The expression agrees with the inspection of Langer and Perline.

Key words: Integrable Hamiltonian system; Vortex filament equation; Hereditary operator; Hamiltonian pair; Master symmetries.

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1 Introduction

The vortex filament equation [1] is a nonlinear evolution equation describing the time development of a very thin vortex tube. The equation is derived from the dynamics of 3-dimensional incompressible fluid with the local induction approximation and is written as

$$\dot{\gamma} = \kappa \mathbf{b}, \tag{1}$$

where γ is the curve of the vortex filament parametrized by the arclength, dot stands for the differential with respect to the time, κ is the curveature of γ, and $\mathbf{b}$ is the bi-normal vector along γ. It is well-known that the vortex filament equation (1) is closely related to the cubic nonlinear Schrödinger (NLS) equation, and the Hasimoto map provides a connection between them [2]. The NLS equation is an infinite-dimensional completely integrable Hamiltonian system [3].

Marsden and Weinstein [4] constructed a Hamiltonian description of the vortex filament equation in their study on the moment map for the action of the unimodular diffeomorphism group of $\mathbf{R}^3$. Langer and Perline [5] introduced the space BAL — the space of balanced asymptotically linear curves (see Section 2) — as a phase space for the system of vortex filament, and showed that the Hasimoto map is a Poisson map from BAL with the Marsden-Weinstein Poisson structure to (a certain equivalence class of) the phase space of the NLS system with the 'fourth' Poisson structure. This result says that the Hasimoto map induces constants of motion in involution for the vortex filament equation as the pull-back of those for the NLS system, hence the vortex filament equation can be understood as a completely integrable system. Further, Langer and Perline found a recursion operator, which generates infinite sequence of commuting Hamiltonian vector fields on BAL.

For some typical integrable Hamiltonian systems, such as the NLS equation, the integrability is studied from various aspects and many remarkable structures are known to exist [3, 6, 7, 8, 9, 10, 11, 12]. It is therefore natural to ask whether the same or similar structures exist for the system of vortex filament. In this paper we focus on structures that are described in the language of differential geometry; we will investigate the hereditary property [9, 11] of the recursion operator, Hamiltonian pair [7, 8], and master symmetries [10]. For these, the answers are all affirmative; the space BAL admits these structures. The asymptotic boundary condition defining BAL is critical for this result; a different situation is encountered when the curve of a vortex filament is supposed to be a loop [13].

The paper is organized as follows: In Section 2, the definition of BAL is clarified. It

involves introducing a further condition to the conditions defining BAL of [5]. Also in this section, some basic notions are described, and several useful formulae for the variational calculus on BAL are summarized. In Section 3, carefully specifing what are admissible vector fields and what are admissible functionals, we define a differential calculus on BAL. The calculus provides the framework for the subsequent analysis. Section 4, consists of two subsections. In Subsection 4.1, a recursion operator is shown to be hereditary, and its several consequences are presented. In Subsection 4.2, it is shown with the help of the hereditary recursion operator that certain two operators form a Hamiltonian pair. In Section 5, master symmetries are investigated and are applied to deriving an expression of constants of motion in involution. This proves the inspection of Langer and Parline.

2 Balanced Asymptotically Linear Curves

Let APC be the space of infinitely extended, arclength-parametrized smooth curves in the Euclidean space $\mathbf{R}^3$ with the standard metric $\langle\ ,\ \rangle$. We imply by the letter γ an element of APC and by s the parameter for it; $s \mapsto \gamma(s)$ is a smooth map $\mathbf{R} \to \mathbf{R}^3$ such that $\partial\gamma(s)/\partial s$ is a unit vector in the tangent space $T_{\gamma(s)}\mathbf{R}^3$.

A map $APC \times \mathbf{R} \to \mathbf{R}$ is referred to as a scalar field on APC. A map $\mathbf{x}\colon APC \times \mathbf{R} \to \coprod T_{\gamma(s)}\mathbf{R}^3$ such that $\mathbf{x}(\gamma,\ s) \in T_{\gamma(s)}\mathbf{R}^3$ is referred to as a tangent field (the term 'vector field' is reserved for the differential calculus). Here, $\coprod$ stands for the direct-sum with respect to the index $(\gamma,\ s) \in APC \times \mathbf{R}$. Similar terminology is used also for a subset BAL, a space that we wish to manifest in this section.

The differential operator with respect to s is denoted by ∂_s (when acting on scalar fields) or by ∇_s (when acting on tangent fields). These operators satisfy

$$\partial_s(fg) = (\partial_s f)g + f\partial_s g, \tag{2}$$

$$\nabla_s(f\mathbf{x}) = (\partial_s f)\mathbf{x} + f\nabla_s\mathbf{x}, \quad \partial_s\langle\mathbf{x},\ \mathbf{y}\rangle = \langle\nabla_s\mathbf{x},\ \mathbf{y}\rangle + \langle\mathbf{x},\ \nabla_s\mathbf{y}\rangle \tag{3}$$

for all scalar fields f, g and tangent fields $\mathbf{x}$, $\mathbf{y}$. As in the equations above, we will often surpress the argument $(\gamma,\ s)$.

A scalar field F is called a functional if F is independent of s, *i.e.*, $\partial_s F = 0$.

We say a scalar field f is asymptotically polynomial-like if there exists a polynomial $P(s) \in \mathbf{R}[s]$ such that $f(\gamma,\ s)/P(s) \to 0$ in the limit $s \to \pm\infty$ for every curve γ. We say a

scalar field f is rapidly-decreasing if $f(\gamma,\, s)P(s)$ for every polynomial $P(s) \in \mathbf{R}[s]$ converges to zero in the limit $s \to \pm\infty$ for every curve γ.

Let f be a scalar field. The scalar field $\partial_s^{-1} f$ (anti-differentiation of f) and the functional $\textstyle\int f$ (definite integration of f) are defined by

$$(\partial_s^{-1} f)(\gamma,\, s) = \frac{1}{2}\Big(\int_{-\infty}^{s} f(\gamma,\, \tilde{s})\, d\tilde{s} - \int_{s}^{\infty} f(\gamma,\, \tilde{s})\, d\tilde{s}\Big), \tag{4}$$

$$\Big(\int f\Big)(\gamma) = \int_{-\infty}^{\infty} f(\gamma,\, \tilde{s})\, d\tilde{s} \tag{5}$$

provided that the integrations in the equations above converge. In emplying operators ∂_s^{-1} and $\textstyle\int$ in the following sections, we will ensure the convergence by introducing certain rules. It is easy to see

$$\partial_s \partial_s^{-1} f = \partial_s^{-1} \partial_s f = f, \quad \partial_s \textstyle\int f = \int \partial_s f = 0, \tag{6}$$

$$\partial_s^{-1}(Ff) = F\partial_s^{-1} f, \quad \textstyle\int (Ff) = F \int f, \tag{7}$$

where f is a rapidly-decreasing scalar field and F is a functional.

We denote by $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ the tangent fields forming the Frénet frame, namely, $\mathbf{t}(\gamma,\, s)$, $\mathbf{n}(\gamma,\, s)$ and $\mathbf{b}(\gamma,\, s)$ are orthonormal vectors in $T_{\gamma(s)}\mathbf{R}^3$ satisfying $\mathbf{t}(\gamma,\, s) = \partial\gamma(s)/\partial s$ and the Frénet-Serret relation

$$\nabla_s \mathbf{t} = \kappa \mathbf{n}, \quad \nabla_s \mathbf{n} = -\kappa \mathbf{t} + \tau \mathbf{b}, \quad \nabla_s \mathbf{b} = -\tau \mathbf{n}. \tag{8}$$

Here, κ and τ are scalar fields characterized by (8), namely, $\kappa(\gamma)$ and $\tau(\gamma)$ are the curvature and torsion, respectively, of the curve γ. Every tangent field is uniquely written as a linear combination of $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ with the coefficients in scalar fields.

The space BAL introduced in [5] is a subset of APC such that (a) the curvature $\kappa(\gamma)$ of $\gamma \in BAL$ is non-vanishing, (b) $\gamma \in BAL$ is asymptotic to a fixed line, *e.g.*, to z-axis, and (c) ambiguity in the parametrization is completely eliminated with imposing balancing condition.

To describe the balancing condition, we need to fix a reference curve $\gamma_0 \in APC$ (or a reference line, z-axis) fulfilling the asymptotic condition as in (b). The condition (b) says the existence of functionals $\ell_\pm \colon BAL \to \mathbf{R}$ with which the asymptotic behaviour of $\gamma \in BAL$ in the region $s \to \pm\infty$ is written as $\gamma(s \pm \ell_\pm(\gamma)) \to \gamma_0(s)$. With these functionals, the balancing condition (c) for $\gamma \in BAL$ can be witten as $\ell_+(\gamma) = \ell_-(\gamma)$. The functional $\ell := \ell_+ + \ell_-$ referred to as the renormalized length (relative to γ_0) is well-defined, though the curve $\gamma \in BAL \subset APC$ is of infinite length.

We supplement the condition (b) with prescribing how $\gamma \in BAL$ converges to the reference curve γ_0; we suppose

$$\begin{cases} \kappa \text{ is rapidly-decreasing, and} \\ \kappa^{-1}\partial_s{}^n\kappa \text{ and } \partial_s{}^n\tau,\ n = 0,\ 1,\ \ldots \text{ are all asymptotically polynomial-like,} \end{cases} \tag{9}$$

where $\kappa^{-1} := 1/\kappa$.

In the next section we introduce a differential calculus on BAL. The action of vector fields on functionals in this calculus is defined to reproduce the usual variational calculus. Here we make a few remarks on the variational calculus and give several useful formulae. For more detailed description, we refer to [5].

Let $\mathbf{x}$ be a tangent field written as

$$\mathbf{x} = \partial_s^{-1}(\kappa g)\mathbf{t} + g\mathbf{n} + h\mathbf{b} \tag{10}$$

with certain rapidly-decreasing scalar fields g, h. Below, $\mathbf{x}(\gamma)$ is identified with a variational vector field along γ. The restriction on $\mathbf{x}$ mentioned above is to force the variation to keep the arclength-parametrization and conditions (b), (c).

In this paper the variational differential operator associated with $\mathbf{x}$ of the form (10) is denoted by $\delta_{\mathbf{x}}$ (when acting on scalar fields) or by $\nabla_{\mathbf{x}}$ (when acting on tangent fields). For calculating the former, the following formulae are useful:

$$\delta_{\mathbf{x}}(fg) = (\delta_{\mathbf{x}} f)g + f\delta_{\mathbf{x}} g, \tag{11}$$

$$\delta_{\mathbf{x}} \partial_s f = \partial_s \delta_{\mathbf{x}} f, \quad \delta_{\mathbf{x}} \partial_s^{-1} f = \partial_s^{-1}\delta_{\mathbf{x}} f, \quad \delta_{\mathbf{x}} \textstyle\int f = \int \delta_{\mathbf{x}} f, \tag{12}$$

$$\delta_{\mathbf{x}} \kappa = \langle \mathbf{n},\ \nabla_s \nabla_s \mathbf{x}\rangle, \tag{13}$$

$$\delta_{\mathbf{x}} \tau = \partial_s\langle \kappa^{-1}\mathbf{b},\ \nabla_s \nabla_s \mathbf{x}\rangle + \langle \kappa\mathbf{b},\ \nabla_s \mathbf{x}\rangle, \tag{14}$$

$$\delta_{\mathbf{x}} s = 0, \tag{15}$$

$$\delta_{\mathbf{x}} \ell = \textstyle\int \langle -\kappa\mathbf{n},\ \mathbf{x}\rangle, \tag{16}$$

where f and g are scalar fields. By abuse of notation, we often use the letter s, by which we mean the scalar field $\hat{s}$ such that $\hat{s}(\gamma,\ s) = s$, as we have done in (15). The latter can be calculated by the following formulae:

$$\nabla_{\mathbf{x}}(f\mathbf{y}) = (\delta_{\mathbf{x}} f)\mathbf{y} + f\nabla_{\mathbf{x}} \mathbf{y}, \tag{17}$$

$$\nabla_{\mathbf{x}} \mathbf{t} = \langle \mathbf{n},\ \nabla_s \mathbf{x}\rangle\mathbf{n} + \langle \mathbf{b},\ \nabla_s \mathbf{x}\rangle\mathbf{b}, \tag{18}$$

$$\nabla_{\mathbf{x}} \mathbf{n} = \langle \kappa^{-1}\mathbf{b},\ \nabla_s \nabla_s \mathbf{x}\rangle\mathbf{b} - \langle \mathbf{n},\ \nabla_s \mathbf{x}\rangle\mathbf{t}, \tag{19}$$

$$\nabla_{\mathbf{x}} \mathbf{b} = -\langle \mathbf{b},\ \nabla_s \mathbf{x}\rangle\mathbf{t} - \langle \kappa^{-1}\mathbf{b},\ \nabla_s \nabla_s \mathbf{x}\rangle\mathbf{n}, \tag{20}$$

where f is a scalar field and $\mathbf{y}$ is a tangent field. These satisfy

$$\delta_{\mathbf{x}} \langle \mathbf{y},\ \mathbf{z} \rangle = \langle \nabla_{\mathbf{x}} \mathbf{y},\ \mathbf{z} \rangle + \langle \mathbf{y},\ \nabla_{\mathbf{x}} \mathbf{z} \rangle. \tag{21}$$

3 Differential Calculus

Let A_n, $n \in \mathbf{Z}$, be the ∂_s-invariant space (*i.e.*, $\partial_s f \in A_n$ for all $f \in A_n$) of scalar fields f such that $\kappa^{-n} f$ is asymptotically polynomial-like. The elements of A_n with $n > 0$ are rapidly-decreasing.

We notice the following properties possessed by A_n:

a1. A_n, $\forall n \in \mathbf{Z}$, is an $\mathbf{R}$-vector space,

a2. $A_n \subset A_{n-1}$ (as $\mathbf{R}$-vector spaces) for all $n \in \mathbf{Z}$,

a3. $A_{-\infty} := A_0 \cup A_{-1} \cup \cdots$ is a commutative associative $\mathbf{R}$-algebra with the unit 1,

a4. $fg \in A_{i+j}$ if $f \in A_i$, $g \in A_j$ for all $i,\ j \in \mathbf{Z}$,

a5. ∂_s is an $\mathbf{R}$-linear operator such that $\partial_s(A_n) \subset A_n$ for all $n \in \mathbf{Z}$,

a6. ∂_s^{-1} and $\int$ are $\mathbf{R}$-linear operators such that $\partial_s^{-1} f \in A_0$ and $\int f \in A_0$ for all $f \in A_2$,

a7. $\kappa \in A_1$, $\kappa^{-1} \in A_{-1}$, and $\tau,\ s,\ \ell,\ 1 \in A_0$,

b1. ∂_s is a derivation of $A_{-\infty}$, *i.e.*, Eq. (2) hold for all $f,\ g \in A_{-\infty}$,

b2. Eqs. (6) and (7) hold for all $f \in A_2$ and $F \in \operatorname{Ker} \partial_s$,

b3. $\int (f \partial_s^{-1} g) = -\int (g \partial_s^{-1} f)$ for all $f,\ g \in A_2$,

b4. $\kappa\kappa^{-1} = \partial_s s = 1$, and $\partial_s \ell = 0$.

Let us consider the objects $\mathcal{E}_n$ that are fully characterized by the rules a1–a7 above; regarding a1–a7 (in which A_n should be read as $\mathcal{E}_n$) as the axioms for $\mathcal{E}_n$, we define $\mathcal{E}_n$, $n \in \mathbf{Z}$, as a family of objects generated by the symbols or indeterminates $\{\kappa,\ \kappa^{-1},\ \tau,\ s,\ \ell,\ 1\}$ with the algebraic operations. Here and in the following paragraph, by algebraic operations we mean addition, scaling by a real number, multiplication, ∂_s, ∂_s^{-1} and $\int$. It is the implication of a6 that ∂_s^{-1} and $\int$ cannot act on $\mathcal{E}_n$, $n < 2$. By definition, rules (such as b1–b4) not following from a1–a7 are not available for $\mathcal{E}_n$.

Let $g_1, \ldots, g_r$ be independent variables running over $\mathcal{E}_{n_1}, \ldots, \mathcal{E}_{n_r}$, respectively. We say $f(g_1, \ldots, g_r)$ is an $\mathcal{E}_n$-valued variable algebraically depending on $g_1, \ldots, g_r$ if $f(g_1, \ldots, g_r)$ is an expression written in terms of $\{g_1, \ldots, g_r, \kappa, \kappa^{-1}, \tau, s, \ell, 1\}$ with use of the algebraic operations and if the rules a1–a7 supplemented with the condition $g_i \in \mathcal{E}_{n_i}$ conclude $f(g_1, \ldots, g_r) \in \mathcal{E}_n$.

We denote by $\mathcal{A}_n$ the space of scalar fields on BAL having an expression that belongs to $\mathcal{E}_n$. It is easy to see that $\mathcal{A}_n$ is a subset of A_n. The statements a1–a7 and b1–b4 remain true even if every A_n is read as $\mathcal{A}_n$. Moreover, these together with the positive-definiteness or at least nondegeneracy of the bi-linear form (35) are all of the fundamental setting we need in constructing the theory developed in this paper.

Proposition 1. *Let Φ be an $\mathbf{R}$-linear map $\mathcal{A}_1 \to \mathcal{A}_0$ induced from an $\mathbf{R}$-linear map $\mathcal{E}_1 \to \int(\mathcal{E}_2)$ in the apparent way. If this map is written as $\Phi(g) = \int f(g)$ with an $\mathcal{E}_2$-valued variable $f(g)$ algebraically depending on $g \in \mathcal{E}_1$, then there exists $h \in \mathcal{A}_1$ with which one can write $\Phi(g) = \int gh \ \forall g \in \mathcal{A}_1$ as an equation in $\mathcal{A}_0$.*

Proof. We note the formulae

$$\int(f\partial_s g) = -\int(g\partial_s f), \quad \int(f\partial_s^{-1} g) = -\int(g\partial_s^{-1} f), \quad \int(f\int g) = \int(g\int f), \tag{22}$$

each of which is vaild as an equation in $\mathcal{A}_0$ if the left-hand side is given as an $\mathcal{E}_0$-valued variable algebraically depending on f and g. From bi-$\mathbf{R}$-linearity of multiplication and $\mathbf{R}$-linearity of ∂_s, ∂_s^{-1} and $\int$, we see the existence of an expression $\Phi(g) = \sum_i \int f_i(g)$ with $f_i(g)$ being $\mathcal{E}_2$-valued variables algebraically depending on $g \in \mathcal{E}_1$ such that no additions are used in the expression of $f_i(g)$. Further, it is possible to suppose g appears in each expression $\int f_i(g)$ only once because of the $\mathbf{R}$-linearity of Φ. For such expressions, it is apparent how to apply successively the formulae (22) to $\int f_i(g)$ to rewrite it into the form $\int gh_i$. This process is justified if one considers the equations in $\mathcal{A}_0$, while consideration in $\mathcal{E}_0$ is useful in verifying that the expressions $\int(\cdots)$ appearing in each step make sense as $\mathcal{E}_0$-valued variables and eventually in deducing $h_i \in \mathcal{A}_1$. $\square$

Let $\mathcal{T}_n$, $n \in \mathbf{Z}$, be the $\mathbf{R}$-vector space of tangent fields defined by $\mathcal{T}_n := \{f\mathbf{t}+g\mathbf{n}+h\mathbf{b} \,|\, f \in \mathcal{A}_{n-1},\ g,\ h \in \mathcal{A}_n\}$. It is easy to see that $\mathcal{T}_n$ is ∇_s-invariant, *i.e.*, $\nabla_s(\mathcal{T}_n) \subset \mathcal{T}_n$.

We denote by φ the surjection associated with the identification $f\mathbf{t} \sim 0$ in $\mathcal{T}_n$, namely, putting $N = \varphi(\mathbf{n})$, $B = \varphi(\mathbf{b})$, we write

$$\varphi(f\mathbf{t} + g\mathbf{n} + h\mathbf{b}) = gN + hB \tag{23}$$

for scalar fields f, g, h. The vector spaces $\varphi(\mathcal{T}_n)$ are left $\mathcal{A}_0$-modules with $f(gN + hB) = (fg)N + (fh)B \ \forall f \in \mathcal{A}_0$, $\forall g,\ h \in \mathcal{A}_n$.

Let $\mathcal{X} := \varphi(\mathcal{T}_1) = \{gN + hB \mid g,\ h \in \mathcal{A}_1\}$. Each element of $\mathcal{X}$ is referred to as a vector field on BAL. Through the injection $\wp\colon \mathcal{X} \to \mathcal{T}_1$ defined by

$$\wp(gN + hB) := \partial_s^{-1}(\kappa g)\mathbf{t} + g\mathbf{n} + h\mathbf{b}, \tag{24}$$

a vector field X induces a derivation — variational differential associated with $\wp(X)$. This derivation acting on $\mathcal{A}_n$ or $\mathcal{T}_n$ can be evaluated with the formulae (11)–(20).

Proposition 2. *The vector spaces $\mathcal{A}_n$ and $\mathcal{T}_n$ are invariant under the action of the vector fields, namely, $\delta_{\wp(X)}(\mathcal{A}_n) \subset \mathcal{A}_n$ and $\nabla_{\wp(X)}(\mathcal{T}_n) \subset \mathcal{T}_n \ \forall X \in \mathcal{X}$, $\forall n \in \mathbf{Z}$.*

Proof. It is essential that every vector field $X = gN + hB$ is written with $g,\ h \in \mathcal{A}_1$. Taking notice of this situation, we find that the formulae (11)–(20) ensure the invariance of $\mathcal{A}_n$ and $\mathcal{T}_n$ under the action of vector fields. □

The space $\mathcal{X}$ of vector fields is a Lie algebra, and $\mathcal{A}_n$ are $\mathcal{X}$-modules. This is an immediate consequence of the following theorem.

Theorem 3. *The $\mathbf{R}$-vector space $\wp(\mathcal{X})$ is a Lie algebra with the product*

$$[\mathbf{x},\ \mathbf{y}] := \nabla_{\mathbf{x}}\mathbf{y} - \nabla_{\mathbf{y}}\mathbf{x} \quad \forall \mathbf{x},\ \mathbf{y} \in \wp(\mathcal{X}). \tag{25}$$

For every $n \in \mathbf{Z}$, the algebra $\mathcal{A}_n$ is a $\wp(\mathcal{X})$-module with the action $\delta_{\mathbf{x}}$, $\mathbf{x} \in \wp(\mathcal{X})$, namely,

$$(\delta_{\mathbf{x}}\delta_{\mathbf{y}} - \delta_{\mathbf{y}}\delta_{\mathbf{x}} - \delta_{[\mathbf{x},\mathbf{y}]})f = 0 \quad \forall \mathbf{x},\ \mathbf{y} \in \wp(\mathcal{X}),\ \forall f \in \mathcal{A}_n. \tag{26}$$

Proof. The statements are verified by using (11)–(20). A convenient procedure is as follows: First, verify that $[\mathbf{x}, \mathbf{y}]$ belongs to $\wp(\mathcal{X})$ for all $\mathbf{x}, \mathbf{y} \in \wp(\mathcal{X})$. Second, show the equation (26) for $f = \kappa, \tau, s, \ell$ and then extend (26) to the whole $\mathcal{A}_{-\infty} = \mathcal{A}_0 \cup \mathcal{A}_{-1} \cup \cdots$. Finally, verify the Jacobi identity in $\wp(\mathcal{X})$ with the help of (26). □

The theorem above is quite similar to Theorem 1 of [13] in particular in the proof, though the considered objects are different.

To simplify expressions, we put

$$\tilde{\nabla}_X := \varphi \circ \nabla_{\wp(X)} \circ \wp \tag{27}$$

for $\forall X \in \mathcal{X}$, so that we can write the commutator product of $\mathcal{X}$ as

$$[X,\, Y] = \tilde{\nabla}_X\, Y - \tilde{\nabla}_Y\, X \quad \forall X,\, Y \in \mathcal{X}. \tag{28}$$

Likewise we put

$$\langle g_1 N + h_1 B,\; g_2 N + h_2 B\rangle_\perp := g_1 g_2 + h_1 h_2, \tag{29}$$

which defines a bi-linear form $\varphi(\mathcal{T}_i) \times \varphi(\mathcal{T}_j) \to \mathcal{A}_{i+j}$. Then, we have

$$\begin{aligned}\tilde{\nabla}_X\, Y = (\delta_{\wp(X)}\, \langle N,\, Y\rangle_\perp) N + (\delta_{\wp(X)}\, \langle B,\, Y\rangle_\perp) B + (\partial_s^{-1} \langle \kappa N,\, Y\rangle_\perp) \varphi \nabla_s\, \wp(X) \\ - \langle \kappa^{-1} B,\, \varphi \nabla_s\, \nabla_s\, \wp(X)\rangle_\perp (\langle B,\, Y\rangle_\perp N - \langle N,\, Y\rangle_\perp B)\end{aligned} \tag{30}$$

for all $X,\, Y \in \mathcal{X}$.

Let $\mathcal{F}$ be the subalgebra of $\mathcal{A}_0$ generated by 1, ℓ and the elements of $\int(\mathcal{A}_2)$. The vector space $\mathcal{F}$ is an $\mathcal{X}$-submodule of $\mathcal{A}_0$, *i.e.*, $\delta_{\wp(X)}\,(\mathcal{F}) \subset \mathcal{F}$ $\forall X \in \mathcal{X}$. We denote the action of $\mathcal{X}$ on $\mathcal{F}$ by the left-action, namely, $XF = \delta_{\wp(X)}\, F$ $\forall X \in \mathcal{X}$, $\forall F \in \mathcal{F}$. This action is a derivation:

$$X(FG) = (XF)G + F(XG) \quad \forall X \in \mathcal{X},\, \forall F,\, G \in \mathcal{F}. \tag{31}$$

Since $\mathcal{F} \subset \mathcal{A}_0$, we see that $\mathcal{X}$ is a left $\mathcal{F}$-module. Taking notice of $\partial_s F = 0$ $\forall F \in \mathcal{F}$ and referring (11)–(20), we easily find $\delta_{\wp(FX)}\, g = F \delta_{\wp(X)}\, g$, $\tilde{\nabla}_{FX}\, Y = F \tilde{\nabla}_X\, Y$ and $\tilde{\nabla}_X\,(FY) = (\delta_{\wp(X)}\, F) Y + F \tilde{\nabla}_X\, Y$ $\forall F \in \mathcal{F}$, $\forall X,\, Y \in \mathcal{X}$, $\forall g \in \mathcal{A}_n$. Hence we see

$$(FX)G = F(XG) \quad \forall F,\, G \in \mathcal{F},\, \forall X \in \mathcal{X}, \tag{32}$$

$$\mathcal{L}_X(FY) = (\mathcal{L}_X F) Y + F \mathcal{L}_X Y \quad \forall X,\, Y \in \mathcal{X},\, \forall F \in \mathcal{F}, \tag{33}$$

where

$$\mathcal{L}_X Y := [X,\, Y], \quad \mathcal{L}_X F := XF \quad \forall X,\, Y \in \mathcal{X},\, \forall F \in \mathcal{F}. \tag{34}$$

Below, we construct in the usual, algebraic manner a differential calculus, in which the algebra $\mathcal{F}$ consisting of functionals on BAL plays the role of the algebra of functions. The construction is based on the pair $(\mathcal{F},\, \mathcal{X})$ of commutative algebra and Lie algebra. It is essential for this construction that $\mathcal{F}$ is a left $\mathcal{X}$-module, $\mathcal{X}$ is a left $\mathcal{F}$-module, and the equations (31)–(33) hold. We would like to make a further remark. Let $g\colon \mathcal{X} \times \mathcal{X} \to \mathcal{F}$ be a symmetric form defined by

$$g(X,\, Y) := \int \langle X,\, Y\rangle_\perp \tag{35}$$

with (29). This is bi-$\mathcal{F}$-linear and positive-definite. We refer to g as the Riemannian structure on BAL. Given $F \in \mathcal{F}$, the vector field $X \in \mathcal{X}$ such that $YF = g(X,\, Y)$ $\forall Y \in \mathcal{X}$ is called

the gradient of F and is denoted by $\operatorname{grad} F$. The existence of the gradient for every element of $\mathcal{F}$ can be verified by virtue of Propositions 1 and 2. In contrast to the differential calculus on finite dimensional Riemannian manifolds, this seems to be quite nontrivial. This situation is necessary for realizing the space of 1-forms as a space identifiable with $\mathcal{X}$.

Let $\mathcal{D}^p$ denote the vector space of maps $\eta\colon \mathcal{X}^{\times p} \to \mathcal{F}$ such that $F := \eta(U_1, \cdots, U_p)$ with $U_i = g_i N + h_i B \in \mathcal{X}$ is $\mathcal{F}$-linear in each U_i, skew-symmetric (if $p \geq 2$) under the exchange of U_i and U_j, $i \neq j$, and F can be expressed as an $\mathcal{E}_0$-valued variable algebraically depending on $g_1, h_1, \ldots, g_p, h_p$. Such a map $\eta \in \mathcal{D}^p$ is referred to as a p-th order differential form or p-form on BAL. From Proposition 1 and the nondegeneracy of (35), we see that for every 1-form ξ there uniquely exists a vector field X such that $\xi(Y) = g(X, Y)$ $\forall Y \in \mathcal{X}$.

The exterior derivative is a map $d\colon \mathcal{D}^p \to \mathcal{D}^{p+1}$ defined by

$$\begin{aligned}(d\eta)(U_0, \ldots, U_p) = &\sum_{i=0}^{p} (-1)^i U_i(\eta(U_0, \ldots, \check{U}_i, \ldots, U_p)) \\ &+ \sum_{i<j} (-1)^{i+j} \eta([U_i, U_j], U_0, \ldots, \check{U}_i, \ldots, \check{U}_j, \ldots, U_p) \\ &\qquad\qquad\qquad \forall \eta \in \mathcal{D}^p,\ \forall U_i \in \mathcal{X},\end{aligned} \tag{36}$$

where $\check{U}_i$ stands for the absense of U_i. This is a coboundary operator, *i.e.*, $d \circ d = 0$. The interior product $\iota_X\colon \mathcal{D}^{p+1} \to \mathcal{D}^p$ with $X \in \mathcal{X}$ is defined by

$$(\iota_X \eta)(U_1, \ldots, U_p) = \eta(X, U_1, \ldots, U_p). \tag{37}$$

The Lie derivative $\mathcal{L}_X$ in the direction of $X \in \mathcal{X}$ is an opeator acting on each $\mathcal{X}$-module consisting of certain $\mathcal{F}$-vectors, so-called tensor fields, and $X \mapsto \mathcal{L}_X$ provides a representation of the Lie algebra $\mathcal{X}$. The definition was already given both on $\mathcal{F}$ and $\mathcal{X}$ in (34). The extension to other tensor fields is made by imposing Leibnitz rule. For example, if R is an $\mathcal{F}$-linear map $\mathcal{X} \to \mathcal{X}$, then $\mathcal{L}_X(RY) = (\mathcal{L}_X R)Y + R\mathcal{L}_X Y$ $\forall X, Y \in \mathcal{X}$. For a p-form η, the formula

$$\mathcal{L}_X \eta = \iota_X d\eta + d\iota_X \eta \tag{38}$$

is available. It is possible to introduce the exterior product, which is, however, not used in this paper.

4 Recursion Operator and Hamiltonian Pair

An $\mathcal{F}$-linear map $K\colon \mathcal{X} \to \mathcal{X}$ is called a skew-adjoint operator if $g(KX,\ Y) = -g(X,\ KY)$. With a skew-adjoint operator J defined by

$$J(X) = \langle B,\ X\rangle_\perp N - \langle N,\ X\rangle_\perp B, \tag{39}$$

the Marsden-Weinstein Poisson structure [4] can be written as $\{F, G\} = (J\,\mathrm{grad}\,F)G\ \forall F, G \in \mathcal{F}$. The operator J, $J^2 = -1$, is a complex structure; it can be shown by a direct calculation that the Nijenhuis torsion (see Eq. (43)) of J vanishes, though this fact is not used in this paper. The vortex filament equation (1) can be understood as a Hamiltonian equation with the Hamiltonian functional ℓ; indeed

$$\kappa\mathbf{b} = \wp(J\,\mathrm{grad}\,\ell). \tag{40}$$

Making use of the Hasimoto map, Langer and Perline [5] found that the vector fields κB, $KJ^{-1}(\kappa B)$, $(KJ^{-1})^2(\kappa B)$, ... are Hamiltonian flows (see Subsection 4.2) associated with the constants of motion in involution, where K is another skew-adjoint operator defined by

$$K(X) = J\varphi\nabla_s\,\wp J(X). \tag{41}$$

The operator

$$R := KJ^{-1} = J \circ \varphi \circ \nabla_s \circ \wp \tag{42}$$

is referred to as the recursion operator. It should be emphasized that the definition of $\mathcal{X}$ given in the preceding section is consistent with J, K and R, namely, these opertors make sense as $\mathcal{F}$-linear maps $\mathcal{X} \to \mathcal{X}$.

Below, after giving several results on the recursion operator R (Subsection 4.1), we shall show that J and K form a Hamiltonian pair (Subsection 4.2). The approach pursued in this section is similar to that of [11].

4.1 The hereditary property

Let R be an $\mathcal{F}$-linear map $\mathcal{X} \to \mathcal{X}$. In most statements of this subsection, we need not suppose R is the recursion operator defined by (42); the exception is Theorem 4.

The Nijenhuis torsion N_R of an $\mathcal{F}$-linear operator $R\colon \mathcal{X} \to \mathcal{X}$ is an $\mathcal{F}$-linear map $\mathcal{X} \times \mathcal{X} \to \mathcal{X}$ defined by

$$\begin{aligned} N_R(X,\ Y) &:= (\mathcal{L}_{RX}R - R\mathcal{L}_X R)Y \\ &= [RX,\ RY] - R[RX,\ Y] - R[X,\ RY] + R^2[X,\ Y]. \end{aligned} \tag{43}$$

An $\mathcal{F}$-linear operator R such that $N_R = 0$ is said to be hereditary [6, 9]; see also [7, 8, 11]. For a hereditary operator R, it is easy to see that $\mathcal{L}_{RX}(R^n) = R\mathcal{L}_X(R^n)$ and further $\mathcal{L}_{R^m X}(R^n) = R^m \mathcal{L}_X(R^n)$, namely,

$$[R^m X,\ R^n Y] - R^n[R^m X,\ Y] = R^m[X,\ R^n Y] - R^{m+n}[X,\ Y] \tag{44}$$

for all vector fields X, Y.

Theorem 4. *The recursion operator R defined by (42) is hereditary.*

Proof. For all $X \in \mathcal{X}$, define $\tilde{\nabla}_X R\colon \mathcal{X} \to \mathcal{X}$ by

$$(\tilde{\nabla}_X R)(Y) = \tilde{\nabla}_X (RY) - R(\tilde{\nabla}_X Y) \quad \forall Y \in \mathcal{X}, \tag{45}$$

which is found to be

$$(\tilde{\nabla}_X R)(Y) = (\partial_s^{-1}\langle \kappa N,\ RY\rangle_\perp) J^{-1} RX + (\partial_s^{-1}\langle RX,\ RY\rangle_\perp)\kappa B \tag{46}$$

by using (30). Substituting the formula above into

$$N_R(X,\ Y) = (\tilde{\nabla}_{RX} R)(Y) - R((\tilde{\nabla}_X R)(Y) - (X \leftrightarrow Y),$$

we find $N_R = 0$. $\square$

Let η be a p-form, $p \geq 1$. We say η is compatible with R if

$$\iota_{RY}\iota_X \eta = \iota_Y \iota_{RX}\eta \quad \forall X,\ Y \in \mathcal{X}. \tag{47}$$

By definition, every 1-form is compatible with R. Suppose η is a p-form compatible with R. Then, it is possible to define a p-form $\eta \circ R$ such that $\iota_X(\eta \circ R) = \iota_{RX}\eta$ for all $X \in \mathcal{X}$. Further, $\eta \circ R$ is again compatible with R. Hence a p-form compatible with R induces p-forms $\eta \circ R^n$ compatible with R, $n = 0,\ 1,\ 2,\ \ldots$, such that

$$\iota_X(\eta \circ R^n) = \iota_{R^n X}\eta \quad \forall X \in \mathcal{X}. \tag{48}$$

For a p-form η compatible with R, we have the formulae

$$\mathcal{L}_{R^n X}\eta = \iota_{R^n X} d\eta + \mathcal{L}_X(\eta \circ R^n) - \iota_X d(\eta \circ R^n), \tag{49}$$

$$\iota_Y \mathcal{L}_X(\eta \circ R) = \iota_{RY}\mathcal{L}_X \eta + \iota_{(\mathcal{L}_X R)Y}\eta, \tag{50}$$

where n is an arbitrary non-negative integer and X is an arbitrary vector field. The former (49) is an immedate consequense of (38). The latter (50) is nothing but the Leibnitz rule

$$(\mathcal{L}_X(\eta \circ R))(Y,\, U_2,\, \ldots,\, U_p) = (\mathcal{L}_X\eta)(RY,\, U_2,\, \ldots,\, U_p) + \eta((\mathcal{L}_X R)Y,\, U_2,\, \ldots,\, U_p).$$

For a 2-form ω compatible with R, we have

$$\begin{aligned} &\{(\mathcal{L}_{R^n X}\omega)(Y,\, Z) - (\omega \circ R^n)(X,\, [Y,\, Z])\} + \mathrm{cycle}(X,\, Y,\, Z) \\ &= \{d\omega(R^n X,\, Y,\, Z) + \mathrm{cycle}(X,\, Y,\, Z)\} - 2d(\omega \circ R^n)(X,\, Y,\, Z), \end{aligned} \tag{51}$$

where n is an arbitrary non-negative integer and X, Y, Z are arbitrary vector fields. The formula (51) is verified as follows: Starting with the substitution of (49), we calculate the left-hand side of (51) as

$$\begin{aligned} &d\omega(R^n X,\, Y,\, Z) + (\mathcal{L}_X(\omega \circ R^n))(Y,\, Z) - d(\omega \circ R^n)(X,\, Y,\, Z) \\ &- (\omega \circ R^n)(X,\, [Y,\, Z]) + \mathrm{cycle}(X,\, Y,\, Z) \\ = &\{d\omega(R^n X,\, Y,\, Z) + (\mathcal{L}_X(\omega \circ R^n))(Y,\, Z) - (\omega \circ R^n)(Y,\, [Z,\, X]) \\ &+ \mathrm{cycle}(X,\, Y,\, Z)\} - 3d(\omega \circ R^n)(X,\, Y,\, Z) \\ = &\{d\omega(R^n X,\, Y,\, Z) + X((\omega \circ R^n))(Y,\, Z)) - (\omega \circ R^n)([X,\, Y],\, Z) \\ &+ \mathrm{cycle}(X,\, Y,\, Z)\} - 3d(\omega \circ R^n)(X,\, Y,\, Z). \end{aligned}$$

Then, recalling the definition (36) of exterior derivative, we see this equals to the right-hand side of (51).

Lemma 5. *Let R be an $\mathcal{F}$-linear map $\mathcal{X} \to \mathcal{X}$. If η is a p-form compatible with R, then*

$$\iota_Y \iota_X d(\eta \circ R^2) - \iota_{RY} \iota_X d(\eta \circ R) - \iota_Y \iota_{RX} d(\eta \circ R) + \iota_{RY} \iota_{RX} d\eta = -\iota_{N_R(X,\,Y)}\eta \tag{52}$$

for all vector fields X, Y.

Proof. That the left-hand side of (52) equals to

$$\iota_Y \mathcal{L}_X(\eta \circ R^2) - \iota_{RY}\mathcal{L}_X(\eta \circ R) - \iota_Y \mathcal{L}_{RX}(\eta \circ R) + \iota_{RY}\mathcal{L}_{RX}\eta$$

can be shown by the substitution of the identity $\iota_Y \iota_X d = \iota_Y \mathcal{L}_X - \mathcal{L}_Y \iota_X + d\iota_Y \iota_X$ $\forall X,\, Y \in \mathcal{X}$ following from (38). Using (50), we can rewrite the expression above into $\iota_{(\mathcal{L}_X R)Y}(\eta \circ R) - \iota_{(\mathcal{L}_{RX} R)Y}\eta$, which obviously equals to the right-hand side of (52). ∎

4.2 Schouten bracket and Hamiltonian pair

Let us recall the notion of a Hamiltonian operator/pair [7, 8].

Let H_i, $i = 0,\ 1$ be $\mathcal{F}$-linear maps $\mathcal{D}^1 \to \mathcal{X}$. Suppose H_i are skew-symmetric, *i.e.*,

$$\xi(H_i\eta) = -\eta(H_i\xi) \quad \forall \xi,\ \eta \in \mathcal{D}^1. \tag{53}$$

The Schouten bracket $[H_0,\ H_1]$ between H_0 and H_1 is a skew-symmetric tri-$\mathcal{F}$-linear map $\mathcal{D}^1 \times \mathcal{D}^1 \times \mathcal{D}^1 \to \mathcal{F}$ defined by

$$[H_0,\ H_1](\xi,\ \eta,\ \zeta) = \xi(H_0\mathcal{L}_{H_1\eta}\zeta) + (H_0 \leftrightarrow H_1) + \mathrm{cycle}(\xi,\ \eta,\ \zeta). \tag{54}$$

A skew-symmetric $\mathcal{F}$-linear map $H_0\colon \mathcal{D}^1 \to \mathcal{X}$ is referred to as a Hamiltonian operator if $[H_0,\ H_0] = 0$. The vector field $H_0 dF$ associated with $F \in \mathcal{F}$ via Hamiltonian operator H_0 is called the Hamiltonian vector field of F. A Hamiltonian operator H_0 induces the Poisson structure

$$\{F,\ G\}_{H_0} = (H_0 dF)G \quad \forall F,\ G \in \mathcal{F} \tag{55}$$

and $H_0 \circ d$ is a morphism $\mathcal{F} \to \mathcal{X}$ of Lie algebras. Hamiltonian operators H_0 and H_1 are said to be a Hamiltonian pair if $[H_0,\ H_1] = 0$.

The existence of a Hamiltonian pair with certain conditions implies the integrability — the existence of a sequence of functionals in involution or Poisson-commutative functionals.

Returning to the case of BAL, we define $H_n\colon \mathcal{D}^1 \to \mathcal{X}$, $n = 0,\ 1,\ \ldots$ by

$$H_n = R^n H_0, \quad g(X,\ H_0\eta) = -\eta(JX) \quad \forall X \in \mathcal{X},\ \forall \eta \in \mathcal{D}^1 \tag{56}$$

with J and R defined by (39) and (42). Under the identification caused by the Riemannian structure g, we see that H_0 and H_1 are nothing but the operators J and K, respectively. Indeed, $H_n \circ d = R^n J \circ \mathrm{grad}$.

We shall show that H_m and H_n form a Hamiltonian pair. For this aim, it is useful to introduce the sequence of 2-forms Ω_n, $n = 0,\ 1,\ \ldots$,

$$\Omega_n = \Omega_0 \circ R^n, \quad \Omega_0(X,\ Y) = g(J^{-1}X,\ Y) \quad \forall X,\ Y \in \mathcal{X}. \tag{57}$$

The well-definedness of Ω_n as 2-forms is explained as follows: As was mentioned, J and K are skew-adjoint operators. From the skew-adjointness of J, we see that Ω_0 is well-defined as a 2-form. From the skew-adjointness of J and K, we see that Ω_0 is compatible with R.

Hence, $\Omega_n = \Omega_0 \circ R^n$ are well-defined as 2-forms. These 2-forms are related to H_n in the following way:

$$\Omega_m(H_n\xi,\, Y) = \xi(R^{m+n}Y) \quad \forall\xi \in \mathcal{D}^1,\ \forall Y \in \mathcal{X}. \tag{58}$$

It is possible to show that $d\Omega_0 = d\Omega_1 = 0$ by using (30). Since R is hereditary, we find

$$d\Omega_0 = d\Omega_1 = d\Omega_2 = \cdots = 0 \tag{59}$$

as a consequence of Lemma 5. We note that $d\Omega_0 = 0$ (and Theorem 6 for the case $m = n = 0$) is implied in [4], because Ω_0 is the symplectic structure corresponding to the Hamiltonian operator J. It should be mentioned that Ω_n, $n \geq 1$, is not the symplectic structure corrsponding to H_n.

Theorem 6. *Two operators arbitrarily chosen from the sequence H_n defined by (56) form a Hamiltonian pair, i.e., $[H_m,\, H_n] = 0$ for all non-negative integers m and n.*

This theorem follows immediately from (59) and the lemma below.

Lemma 7. *Let $R\colon \mathcal{X} \to \mathcal{X}$ be a hereditary operator and Ω_0 a 2-form compatible with R. Suppose the map $\mathcal{X} \to \mathcal{D}^1$, $X \mapsto \iota_X\Omega_0$ is invertible, so that an $\mathcal{F}$-linear map $H_0\colon \mathcal{D}^1 \to \mathcal{X}$ is defined by $\Omega_0(H_0\xi,\, Y) = \xi(Y)$ $\forall\xi \in \mathcal{D}^1$, $\forall Y \in \mathcal{X}$. Then, $H_n := R^n H_0$, $n = 0,\ 1,\ 2,\ \ldots$ are skew-symmetric $\mathcal{F}$-linear maps $\mathcal{D}^1 \to \mathcal{X}$ and the Schouten brackets between them are written as*

$$\begin{aligned} &[H_m,\, H_n](\xi,\, \eta,\, \zeta) \\ &= 4d\Omega_{m+n}(H_0\xi,\, H_0\eta,\, H_0\zeta) \\ &\quad - \{d\Omega_m(H_n\xi,\, H_0\eta,\, H_0\zeta) + (m \leftrightarrow n) + \mathrm{cycle}(\xi,\, \eta,\, \zeta)\} \end{aligned} \tag{60}$$

with $\Omega_n := \Omega_0 \circ R^n$.

Proof. The $\mathcal{F}$-linearity of H_n is apparent. Further, the calculation

$$\begin{aligned} \xi(H_n\eta) &= \Omega_0(H_0\xi,\, H_n\eta) = (\Omega_0 \circ R^n)(H_0\xi,\, H_0\eta) \\ &= -(\Omega_0 \circ R^n)(H_0\eta,\, H_0\xi) = -\Omega_0(H_0\eta,\, H_n\xi) = -\eta(H_n\xi) \end{aligned}$$

shows that H_n are skew-symmetric. The Schouten bracket $[H_m,\, H_n]$ is therefore well-defined and is calculated as follows: First, we notice

$$\xi(H_m\mathcal{L}_{H_n\eta}\zeta) = -(\mathcal{L}_{H_m\eta}\zeta)(H_n\xi) = -(H_m\eta)(\zeta(H_n\xi)) + \zeta([H_m\eta,\, H_n\xi]),$$

so that we have

$$[H_m,\ H_n](\xi,\ \eta,\ \zeta) = \{ - (H_m\eta)(\zeta(H_n\xi)) + \zeta([H_m\eta,\ H_n\xi])\} + (m \leftrightarrow n) + \mathrm{cycle}(\xi,\ \eta,\ \zeta).$$

Using (44), we see

$$\begin{aligned}[H_m,\ H_n](\xi,\ \eta,\ \zeta) = \{ &- (H_m\eta)(\Omega_n(H_0\zeta,\ H_0\xi)) + \Omega_n(H_0\zeta,\ [H_m\eta,\ H_0\xi]) \\ &+ \Omega_m(H_0\zeta,\ [H_0\eta,\ H_n\xi]) - \Omega_{m+n}(H_0\zeta,\ [H_0\eta,\ H_0\xi])\} \\ &+ (m \leftrightarrow n) + \mathrm{cycle}(\xi,\ \eta,\ \zeta).\end{aligned}$$

Taking notice of the symmetry under the exchange $m \leftrightarrow n$ and the cyclic permutation, we see that the first three terms in the right-hand side sum up to $(-\mathcal{L}_{H_m\xi}\Omega_n)(H_0\eta,\ H_0\zeta)$. Then, with the help of (51) we obtain (60). □

5 Symmetries and Master Symmetries

Let X_n, $n = 0,\ 1,\ 2,\ \ldots$ be the vector fields

$$X_n = R^n(\kappa B) \tag{61}$$

and Y_n, $n = 1,\ 2,\ 3,\ \ldots$ the vector fields

$$Y_n = R^{n-1}(s\kappa B), \tag{62}$$

where R is the recursion operator defined by (42). The vector fields X_n are those given in [5] with a difference in their index (shifted by 2).

Lemma 8. *The vector fields X_0 and Y_1 act on the recursion operator R of (42) as follows:*

$$\mathcal{L}_{X_0} R = 0, \quad \mathcal{L}_{Y_1} R = -R^2. \tag{63}$$

Proof. This is shown through a somewhat tedious calculation. It is easy to see $(\mathcal{L}_X R)(Y) = (\tilde{\nabla}_X\, R)(Y) - \tilde{\nabla}_{RY}\, X + R\tilde{\nabla}_Y\, X$ for all vector fields X, Y. We continue the calculation with substitution of (30) and (46), and finally arrive at the lemma. □

As a corollary of the lemma, it is immediate to see

$$\mathcal{L}_{X_0} R^n = 0, \quad \mathcal{L}_{Y_1} R^n = -nR^{n+1}. \tag{64}$$

Proposition 9. *The vector fields $X_0,\ X_1,\ \ldots$ and $Y_1,\ Y_2,\ \ldots$ form a Lie subalgebra of $\mathcal{X}$ such that*

$$[X_n,\ X_m] = 0, \tag{65}$$

$$[X_n,\ Y_m] = (n+2)X_{n+m}, \tag{66}$$

$$[Y_n,\ Y_m] = (n-m)Y_{n+m}. \tag{67}$$

Proof. As was stated in Theorem 4 the recursion operator R is hereditary, hence the formula (44) is available. This formula can be written also in the following form:

$$[R^m X,\ R^n Y] = -R^n(\mathcal{L}_Y R^m)X + R^m(\mathcal{L}_X R^n)Y + R^{m+n}[X,\ Y] \quad \forall X,\ Y \in \mathcal{X}.$$

Substituing (64) to the formula above, we obtain (65) and (67). We can show (66) in much the same way with the help of

$$[X_0,\ Y_1] = 2X_1. \tag{68}$$

This equation can be verified by using (30). ∎

It is immediate from (1) to see that X_0 is the flow of the vortex filament equation. A vector field commuting with the flow X_0 of an evolution equation is, generally, called a symmetry (of the equation). Thus X_n of (61) can be described as symmetries of the vortex filament equation. Since these symmetries are generated from X_0 by the action of Y_n as in (66), vector fields Y_n are referred to as master symmetries [10]. Sometimes the term 'recursion operator' is used for meaning an $\mathcal{F}$-linear operator $R\colon \mathcal{X} \to \mathcal{X}$ such that $\mathcal{L}_{X_0} R = 0$, *i.e.*, an operator that maps a symmetry into another symmetry [6]. The operator R of (42) is a recursion operator also in this sense.

Proposition 10. *By means of the Lie derivative, symmetries (61) and master symmetries (62) act on the 2-forms Ω_n defined by (57) as follows:*

$$\mathcal{L}_{X_{m-1}}\Omega_n = 0, \tag{69}$$

$$\mathcal{L}_{Y_m}\Omega_n = (3-m-n)\Omega_{m+n}, \tag{70}$$

where $m = 1,\ 2,\ \ldots$ and $n = 0,\ 1,\ \ldots$.

Proof. Making use of (30), we can show

$$\mathcal{L}_{X_0}\Omega_0 = 0, \quad \mathcal{L}_{Y_1}\Omega_0 = 2\Omega_1 \tag{71}$$

with a somewhat tedious calculation. Using (64), the equation above and Leibnitz rule, we can derive Eqs. (69) and (70) for the case $m = 1$. Further, we see $\mathcal{L}_{R^m X}\Omega_n = \mathcal{L}_X\Omega_{m+n}$ $\forall X \in \mathcal{X}$ as a specific case of (49), since $d\Omega_n = 0$ as in (59). □

Since Ω_n are closed 2-forms, the proposition says

$$d\iota_{X_{m-1}}\Omega_n = 0, \tag{72}$$

$$d\iota_{Y_m}\Omega_n = (3 - m - n)\Omega_{m+n}. \tag{73}$$

Apparently, Ω_n, $n = 1,\ 2,\ 4,\ 5,\ \ldots$ are exact.

Let ζ_n, $n = 0,\ 1,\ \ldots$ be the 1-forms

$$\zeta_n := \iota_{X_n}\Omega_0 = (\iota_{X_0}\Omega_0) \circ R^n. \tag{74}$$

As expressed in (72), 1-forms ζ_n are closed. Since we already know the Lie derivative of X_n and Ω_0 in the direction of X_{m-1} and Y_m, we can easily verify that

$$\mathcal{L}_{X_{m-1}}\zeta_n = 0, \tag{75}$$

$$\mathcal{L}_{Y_m}\zeta_n = (1 - m - n)\zeta_{m+n}, \tag{76}$$

where $m = 1,\ 2,\ \ldots$ and $n = 0,\ 1,\ \ldots$.

Theorem 11. *The 1-forms ζ_n, $n = 0,\ 2,\ 3,\ \ldots$ defined by (74) are exact and are written as $\zeta_0 = dI_0$, $I_0 := \ell$ and*

$$\zeta_n = dI_n, \quad I_n := \frac{1}{1-n}\,\zeta_{n-1}(Y_1), \quad n = 2,\ 3,\ \ldots. \tag{77}$$

The functionals I_n, $n = 0,\ 2,\ 3,\ \ldots$ are in involution with respect to the Poisson bracket associated with H_k with arbitrary k. The vector fields X_n, $n = 0,\ 2,\ 3,\ \ldots$ of (61) are Hamiltonian vector fields of I_n with respect to H_0, namely, $X_n = J\,\mathrm{grad}\,I_n$.

Proof. Since ζ_n are closed 1-forms, (76) can be written as $d\iota_{Y_m}\zeta_n = (1 - m - n)\zeta_{m+n}$, from which we see that ζ_n, $n = 2,\ 3,\ \ldots$ are exact 1-forms written as in (77). The same is easy for the case $n = 0$. Using (55) and (58), we see

$$\begin{aligned}\{I_i,\ I_j\}_{H_k} &= dI_j(H_k dI_i) = \zeta_j(H_k\zeta_i) = \Omega_0(X_j,\ H_k\zeta_i)\\ &= -\zeta_i(R^k X_j) = -\Omega_{i+j+k}(X_0,\ X_0) = 0,\end{aligned}$$

namely, the functionals I_n are in involution with respect to the Poisson bracket associated with H_k. Using (57), we see

$$\Omega_0(J\,\mathrm{grad}\,I_n,\, Y) = g(\mathrm{grad}\,I_n,\, Y) = dI_n(Y) = \zeta_n(Y) = \Omega_0(X_n,\, Y)$$

for all $Y \in \mathcal{X}$. This implies $X_n = J\,\mathrm{grad}\,I_n$. □

Although we gave it in the proof, an explanation for the statements of the theorem other than $\zeta_n = dI_n$ can be found in the general theory [7, 8].

The missing piece, functional I_1 such that $X_1 = J\,\mathrm{grad}\,I_1$ does not exist within $\mathcal{F}$ (the possible candidate for I_1 in other treatments is the total torsion [5, 13]).

The theorem above proves the inspection of Langer and Perline [5] saying

$$I_n = \frac{1}{n-1}\int_{-\infty}^{\infty} ds\, \partial_s^{-1}\langle \kappa N,\, X_{n-1}\rangle_\perp. \tag{78}$$

Partly, the same is given in [13]. The proof is as follows: Inserting $\partial_s s = 1$ into the integrand and then making partial integration, we see that the right-hand side of the equation above equals to

$$\frac{1}{n-1}\int_{-\infty}^{\infty} ds\, \langle -s\kappa N,\, X_{n-1}\rangle_\perp = \frac{1}{n-1}\, g(J^{-1}(s\kappa B),\, X_{n-1}) = \frac{1}{n-1}\,\Omega_0(Y_1,\, X_{n-1}),$$

which obviously coinides with (77). The surface term in this partial integration is absent, since $\partial_s^{-1}\langle \kappa N,\, X_{n_1}\rangle_\perp \in \mathcal{A}_2$ follows from $\langle \kappa N,\, X_{n_1}\rangle_\perp \in \mathcal{A}_2$ by virtue of the fact [5] that $\partial_s^{-1}\langle \kappa N,\, X_{n-1}\rangle_\perp$ can be written as polynomials in ${\partial_s}^n\kappa$ and ${\partial_s}^n\tau$.

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